An Accurate Spatial Differencing Scheme for a Three-Dimensional Full Potential Equation

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Abstract

An accurate spatial differencing scheme is developed for a three-dimensional full potential equation in strong conservation law form. In numerical analysis, this scheme guarantees high accuracy in any irregular three-dimensional curvilinear grids with rapid stretching, skewness, and singularities. This is verified by comparisons of the numerical results with some three-dimensional analytical solutions developed by the authors.

Contents

In the numerical analysis of internal flow, due to the complexity of the flow passage, the governing equation is always transformed from physical space (Cartesian or cylindrical coordinates) into computational space. Therefore, the metric tensor, which is determined only by the geometrical characteristics of physical and computational space, is naturally drawn into the govening equation and the expressions of other quantities.

Because the metric quantities consist of several partial derivatives that must be evaluated by some kind of differencing scheme in the numerical analysis, the method should be consistent with the differencing scheme for the governing equation. Additionally, in physical space, grids are usually formed with rapid stretching, skewness, and singularities. The existence of these geometrical irregularities also affects the values of the metric quantities and, consequently, the numerical results. This means that a good differencing scheme should have a high suitability for any irregular curvilinear grids. These facts were first noticed for the Euler equation by Thomas and Lombard[1] who developed the geometrical conservation law (GCL). Hindman[2] had analyzed the geometrically induced errors for the Euler equation. For the two-dimensional full potential equation, the consistency of differencing schemes has been discussed by Flores, et al.[3] The aim of the present research is to investigate the three-dimensional full potential equation and to find an accurate differencing scheme suitable for any irregular curvilinear grids.

Governing Equation and Differencing Schemes

Based on the assumption that flow is steady and irrotational, the three-dimensional full potential equation can be written in strong conservation law form. In each item of the governing equation, there are three subitems: the metric tensor, density, and velocity component. In the present research, the governing equation is expressed in terms of an arbitrary nonorthogonal curvilinear coordinate system (x^1, x^2, x^3) and corresponding nonorthogonal velocity components.[4]

In numerical analysis, the governing equation is discreted by the standard second-order-accurate central differencing scheme. This means that for flux calculation, the three subitems in the difference equation of the governing equation must be calculated at each midpoint $(i+\frac{1}{2},j,k)$, $(i,j+\frac{1}{2},k)$, and $(i,j,k+\frac{1}{2})$. Moreover, the partial derivatives Φ_{x^1}, Φ_{x^2}, Φ_{x^3} and the metric quantities are also needed in the density calculation at each midpoint. It is clear that these difference operations within the density calculation may be carried out at each midpoint just like the flux calculation or may be performed first at another location such as a mesh point (i,j,k) or cell center $(i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2})$ by a certain differencing scheme and then evaluated at the midpoint by some averaging operation. Furthermore, the two types of derivatives, both for density and flux (i.e., the Φ_{x^1}, Φ_{x^2}, Φ_{x^3} and the partial derivatives in the metric tensor) may also be treated at different locations by using different schemes. Among the possible choices, three have been selected by following two principles: 1) the differencing schemes used to calculate each corresponding item of density and flux in each spatial direction are identical, and 2) the Φ_{x^1}, Φ_{x^2}, Φ_{x^3} and partial derivatives in the metric quantities both for density and flux are calculated at the same location using the same differencing scheme.

For all three schemes, Φ_{x^1}, Φ_{x^2}, and Φ_{x^3} are evaluated at each midpoint for both density and flux by:

$$\left(\frac{\partial\phi}{\partial x^1}\right)_{i+\frac{1}{2},j,k} = \phi_{i+1,j,k} - \phi_{i,j,k} \tag{1a}$$

$$\left(\frac{\partial\phi}{\partial x^2}\right)_{i+\frac{1}{2},j,k} = \frac{1}{4}(\phi_{i+1,j+1,k} + \phi_{i,j,+1,k} - \phi_{i+1,j-1,k} - \phi_{i,j-1,k}) \tag{1b}$$

$$\left(\frac{\partial\phi}{\partial x^3}\right)_{i+\frac{1}{2},j,k} = \frac{1}{4}(\phi_{i+1,j,k+1} + \phi_{i,j,k+1} - \phi_{i+1,j,k-1} - \phi_{i,j,k-1}) \tag{1c}$$

Similar formulas at $(i,j+\frac{1}{2},k)$ and $(i,j,k+\frac{1}{2})$ are developed. The partial derivatives in the metric tensor, however, are treated at different locations. In scheme 1, they are first calculated at each mesh point (i,j,k) by

$$\left(\frac{\partial y}{\partial x^1}\right)_{i,j,k} = \frac{1}{2}(y_{i+1,j,k} - y_{i-1,j,k}) \tag{2a}$$

$$\left(\frac{\partial y}{\partial x^2}\right)_{i,j,k} = \frac{1}{2}(y_{i,j+1,k} - y_{i,j-1,k}) \tag{2b}$$

$$\left(\frac{\partial y}{\partial x^3}\right)_{i,j,k} = \frac{1}{2}(y_{i,j,k+1} - y_{i,j,k-1}) \tag{2c}$$

where y represents the coordinates in physical space. Then, their values at each midpoint are obtained by a simple averaging operation. For scheme 2, these derivatives are first calculated at each cell center $(i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2})$ by

$$\left(\frac{\partial y}{\partial x^1}\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{4}(y_{i+1,j+1,k+1} + y_{i+1,j,k+1} + y_{i+1,j+1,k} + y_{i+1,j,k} - y_{i,j+1,k+1} - y_{i,j,k+1} - y_{i,j+1,k} - y_{i,j,k}) \tag{3a}$$

$$\left(\frac{\partial y}{\partial x^2}\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{4}(y_{i+1,j+1,k+1} + y_{i,j+1,k+1} + y_{i+1,j+1,k} + y_{i,j+1,k} - y_{i+1,j,k+1} - y_{i,j,k+1} - y_{i+1,j,k} - y_{i,j,k}) \tag{3b}$$

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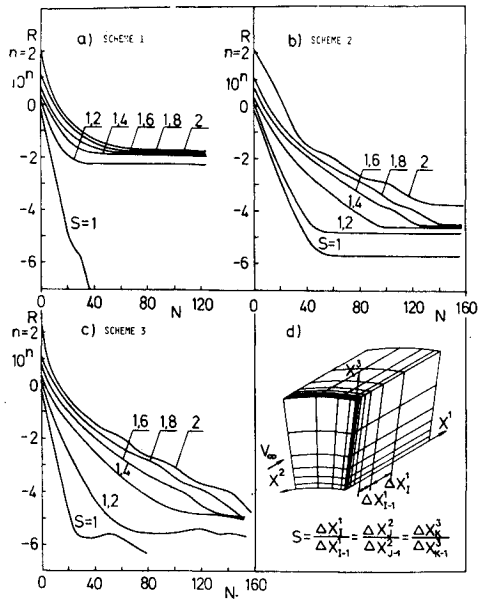

Fig. 1 Maximum residual convergence history comparison in three-dimensional straight-line channel with stretching ratio s.

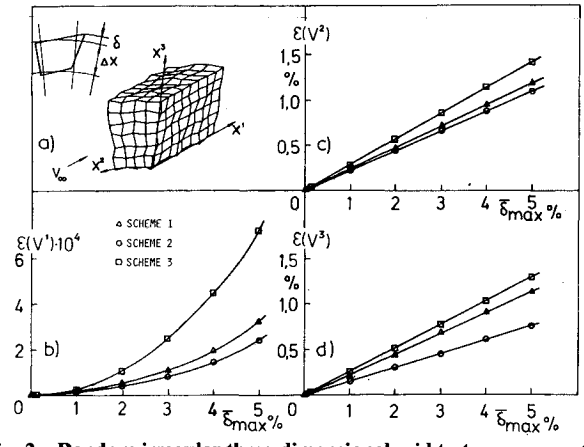

Fig. 2 Random irregular three-dimensional grid test, square-root-of-errors of velocity components V^1, V^2, and V^3 vs maximum mesh point displacement.

$$\left(\frac{\partial y}{\partial x^3}\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{4}(y_{i+1,j+1,k+1} + y_{i,j+1,k+1} + y_{i+1,j,k+1} + y_{i,j,k+1} - y_{i+1,j+1,k} - y_{i,j+1,k} - y_{i+1,j,k} - y_{i,j,k}) \quad (3c)$$

and then evaluated at each midpoint using a simple averaging operation also. For scheme 3, the above derivatives are directly calculated at each midpoint by using Eqs. (1), but the variable y is substituted for ϕ.

Numerical Results

Three analytical solutions for three-dimensional potential flow[5] are used as test problems to examine the accuracy reached by each scheme. These are: uniform straight-line flow with potential $\phi = V_\infty z$, oblique channel flow with potential $\phi = \theta z$, and twisted channel flow similar to the flow in axial turbomachinery cascades with potential $\phi = \sin(n\ell nr)e^{n\theta}(az+b) + m\theta z$.

Figure 1 shows the convergence history of the maximum residual in the first test problem. It can be seen that, when the grid stretching ratio s (defined in Fig. 1d) varies at 1-2, the behavior of scheme 1 greatly deteriorates, but for schemes 2

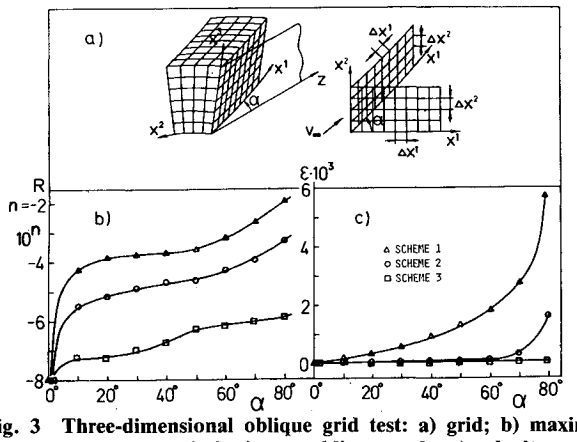

Fig. 3 Three-dimensional oblique grid test: a) grid; b) maximum residual convergence limitation vs oblique angle; c) velocity square-root-of-errors vs oblique angle.

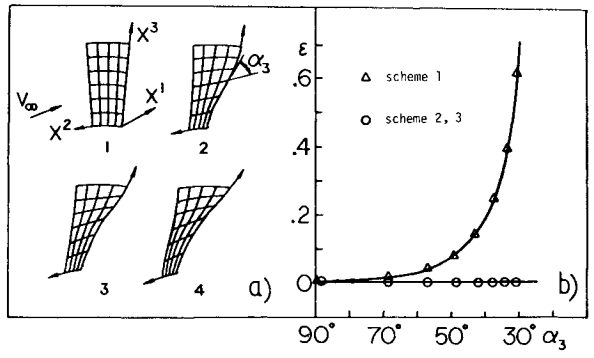

Fig. 4 Three-dimensional twisted grid test: a) grid; b) maximum velocity square-root-of-errors vs twist angle.

and 3 the convergence momentum to high accuracy is still maintained. In Fig. 2, a discontinuous grid test is constructed. This figure demonstrates that scheme 2 is the best one. The scheme most sensitive to this kind of geometrical irregularity, nonsmoothness, is scheme 3. The results for the later two test problems shown in Fig. 3 and 4 indicate that schemes 2 and 3 are very satisfactory and that the behavior of scheme 1 is unacceptable for numerical analysis of a twisted grid.

Conclusions

In order to guarantee high accuracy of numerical results for three-dimensional full potential equation in any irregular curvilinear grid, schemes 2 and 3 are the suitable candidates. But scheme 3 is rather sensitive to geometrical singularities, so for the three-dimensional internal flow calculations in which the grids are usually generated by some simple algebraic technique (without explicitly giving the mathematical relations between physical and computational space), scheme 2 is recommended as the most suitable.

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